Anomalous time correlation in two-dimensional driven diffusive systems

Takenobu Nakamura,* Michio Otsuki,† and Shin-ichi Sasa‡
Department of Pure and Applied Sciences, University of Tokyo, Komaba, Tokyo 153-8902, Japan
(Dated: February 2, 2008)

We study the time correlation function of a density field in two-dimensional driven diffusive systems within the framework of fluctuating hydrodynamics. It is found that the time correlation exhibits power-law behavior in an intermediate time regime in the case that the fluctuation-dissipation relation is violated and that the power-law exponent depends on the extent of this violation. We obtain this result by employing a renormalization group method to treat a logarithmic divergence in time.

PACS numbers: 05.40.-a,05.10.Cc

I. INTRODUCTION

The anomalous time correlation of hydrodynamic modes has been studied for a long period. For an equilibrium fluid, it is understood that this anomaly arises from nonlinear mode coupling effects [1]. By contrast, there is no systematic understanding of the time correlation in nonequilibrium steady states (NESSs) far from local equilibrium. In particular, it is not known how violation of the fluctuation-dissipation relation (FDR) influences the time correlation.

As the simplest example realizing a NESS far from local equilibrium, we consider a two-dimensional driven diffusive system, in which a fluctuating density field is driven locally by an external force. Such a system can be realized in laboratory experiments [2]. Perhaps the simplest model for a theoretical study of the long time behavior in the driven diffusive system is a stochastic differential equation consisting of terms representing a drift due to the external force, diffusion and random noise.

The time correlation function for such a stochastic model has been calculated by employing mode coupling theory [3]. However, the model analyzed in Ref. [3] does not exhibit the long-range spatial correlation that is a generic feature of NESSs in driven diffusive systems of $d \geq 2$ dimensions. The reason that long-range correlation does not appear in that model is that violation of the FDR is not taken into account. Indeed, it is known that, in general, long-range correlation cannot exist when the FDR holds. By contrast, it has been found that the long-range correlation of driven diffusive systems can be described by a linear model with the violation of the FDR [4].

With the above considerations, in the present paper, we study a nonlinear model in which the FDR can be violated. We demonstrate that as a result of this violation, the time correlation is qualitatively altered. Specifically, by employing a perturbative renormalization group (RG) method that treats a logarithmic divergence in time [5], we obtain an expression for the time correlation function. From this expression, we find that power-law behavior appears in the time correlation if and only if the FDR is violated and that the power-law exponent depends on the extent of the violation.

*Electronic address: soushin@jiro.c.u-tokyo.ac.jp
†Electronic address: otsuki@jiro.c.u-tokyo.ac.jp
‡Electronic address: sasa@jiro.c.u-tokyo.ac.jp

II. MODEL

We consider the time evolution of a fluctuating density field $\rho(\boldsymbol{x},t)$ in a two-dimensional space, under the influence of an external driving force in one direction, say the x_1 direction, where $\boldsymbol{x} = (x_1, x_2)$. Note that we study NESSs in the high temperature regime, far from the critical point. We now describe the model we study. First, the conserved quantity ρ obeys the continuity equation

$$\frac{\partial \rho(\boldsymbol{x},t)}{\partial t} + \sum_{i=1}^{2} \frac{\partial J_i(\boldsymbol{x},t)}{\partial x_i} = 0. \quad (1)$$

We assume that the i-th component of the density current, $J_i(\boldsymbol{x},t)$, is given by

$$J_i(\boldsymbol{x},t) = -D_i \partial_i \rho(\boldsymbol{x},t) + \delta_{i1} \bar{J}(\rho(\boldsymbol{x},t)) + \xi_i(\boldsymbol{x},t). \quad (2)$$

Here, the functional form of $\bar{J}$ is such that, with $\bar{\rho}$ the average density, $\bar{J}(\bar{\rho})$ is the average current along the x_1 direction in the steady state. We then approximate $\bar{J}(\rho(\boldsymbol{x},t))$ in the form

$$\bar{J}(\rho(\boldsymbol{x},t)) \approx \bar{J}(\bar{\rho}) + c(\bar{\rho})\delta\rho(\boldsymbol{x},t) + \lambda(\bar{\rho})(\delta\rho(\boldsymbol{x},t))^2, \quad (3)$$

where $\rho(\boldsymbol{x},t) = \bar{\rho} + \delta\rho(\boldsymbol{x},t)$. The term $\xi_i(\boldsymbol{x},t)$ in (2) represents a random current constituting zero mean Gaussian white noise, with

$$\langle \xi_i(\boldsymbol{x},t)\xi_j(\boldsymbol{x}',t')\rangle = 2B_i\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \quad (4)$$

Note that because anisotropy in both the diffusion and noise intensity is expected to arise through effects of the external driving, the diffusion constant, D_i, and the noise intensity, B_i, are assumed to be anisotropic, in general.

Let us simplify the model given above. First, note that the first term on the right-hand side of (3) does

not contribute to the time evolution of the density, and the second term can be eliminated when we study density fluctuations in a frame moving with the velocity c given in (3). To make this explicit, we define the density $\phi(\boldsymbol{x},t) \equiv \delta\rho(\boldsymbol{x}+c\boldsymbol{e}_1 t,t)$, where $\boldsymbol{e}_1$ is the unit vector in the x_1 direction. Furthermore, introducing the parameters χ and Δ, we rewrite B_1 and B_2 as

$$B_i = D_i \chi (1-(-1)^i \Delta). \tag{5}$$

Thus, Δ corresponds to the extent of the violation of the FDR of the second kind [6]. Then, replacing x_i by $\sqrt{D_i}x_i$, ϕ by $\sqrt{\chi}(D_1D_2)^{1/4}\phi$ and ξ_i by $\sqrt{\chi D_i}(D_1D_2)^{-1/4}\xi_i$, we obtain the following dimensionless form of the equation for ϕ:

$$\frac{\partial\phi(\boldsymbol{x},t)}{\partial t} = \sum_{i=1}^{2}\left[\frac{\partial^2\phi(\boldsymbol{x},t)}{\partial x_i^2} - \frac{\partial\xi_i(\boldsymbol{x},t)}{\partial x_i}\right] - \bar{\lambda}\frac{\partial\phi(\boldsymbol{x},t)^2}{\partial x_1}. \tag{6}$$

Here,

$$\langle\xi_i(\boldsymbol{x},t)\xi_j(\boldsymbol{x}',t')\rangle = 2\delta_{ij}(1-(-1)^i\Delta)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'), \tag{7}$$

and $\bar{\lambda}$ is a dimensionless constant given by

$$\bar{\lambda} = \lambda(D_1^3 D_2)^{-1/4}\chi^{1/2}. \tag{8}$$

The renormalization group flow of $(\bar{\lambda},\Delta)$ for the model (6) with (7) is studied in Ref. [7]. Also, the time correlation function has been calculated in the special cases that $\Delta = 0$ (using the mode coupling equation) [3] and $\bar{\lambda} = 0$ [4]. However, as far as we know, the time correlation function for the nonlinear model (6) with the anisotropic noise intensity (7) has never been investigated.

In the analysis below, employing a perturbative expansion with respect to $\bar{\lambda}$ and Δ, we calculate the time correlation function $\hat{C}(\boldsymbol{k},t)$ defined by

$$(2\pi)^2\hat{C}(\boldsymbol{k},t)\delta(\boldsymbol{k}+\boldsymbol{k}') = \left\langle\hat{\phi}(\boldsymbol{k},0)\hat{\phi}(\boldsymbol{k}',t)\right\rangle. \tag{9}$$

Here and below, for an arbitrary function $f(\boldsymbol{x},t)$, we define $\hat{f}(\boldsymbol{k},t)$ by

$$\hat{f}(\boldsymbol{k},t) \equiv \int d^2\boldsymbol{x} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x},t). \tag{10}$$

From the definition (9) and the symmetry of the steady state with respect to translation in time, the equality $\hat{C}(\boldsymbol{k},t) = \hat{C}(\boldsymbol{k},-t)$ holds. Therefore, we consider only $\hat{C}(\boldsymbol{k},t)$ with $t \geq 0$.

III. ANALYSIS

First, we fix Δ and consider the expansion of $\hat{\phi}(\boldsymbol{k},t)$ in $\bar{\lambda}$:

$$\hat{\phi}(\boldsymbol{k},t) = \hat{\phi}^{(0)}(\boldsymbol{k},t) + \bar{\lambda}\hat{\phi}^{(1)}(\boldsymbol{k},t) + \bar{\lambda}^2\hat{\phi}^{(2)}(\boldsymbol{k},t) + \cdots. \tag{11}$$

Substituting (11) into (6) with (10) and extracting all terms proportional to $\bar{\lambda}^n$, we obtain a linear differential equation for $\hat{\phi}^{(n)}$ containing all lower order $\hat{\phi}^{(k)}$ and $\hat{\xi}_i(\boldsymbol{k},t)$. Solving these differential equations under initial conditions set at $t = -\infty$, we can iteratively derive expressions for $\hat{\phi}^{(0)}$, $\hat{\phi}^{(1)}, \cdots$. We then substitute these results into (9). In this way, the correlation function $C(\boldsymbol{k},t)$ is calculated in the form

$$\hat{C}(\boldsymbol{k},t) = \hat{C}^{(0)}(\boldsymbol{k},t) + \bar{\lambda}\hat{C}^{(1)}(\boldsymbol{k},t) + \bar{\lambda}^2\hat{C}^{(2)}(\boldsymbol{k},t) + \cdots. \tag{12}$$

It turns out that it is simplest to obtain the terms $\hat{C}^{(n)}(\boldsymbol{k},t)$ in the above expansion of $\hat{C}(\boldsymbol{k},t)$ by first deriving the terms $\tilde{C}^{(n)}(\boldsymbol{k},\omega)$ in the analogous expansion of $\tilde{C}(\boldsymbol{k},\omega)$, the Fourier transform with respect to time of $\hat{C}(\boldsymbol{k},t)$, and then taking the inverse Fourier transform of these.

The lowest-order contribution to $\hat{C}(\boldsymbol{k},t)$ can be easily calculated as

$$\hat{C}^{(0)}(\boldsymbol{k},t) = \left(1+\Delta\frac{k_1^2-k_2^2}{|\boldsymbol{k}|^2}\right)\mathrm{e}^{-|\boldsymbol{k}|^2 t}. \tag{13}$$

Note that the spatial correlation function, obtained through the Fourier transformation of $\hat{C}^{(0)}(\boldsymbol{k},0)$, exhibits power-law decay of the type $1/r^2$, unless $\Delta = 0$. This illustrates the long-range correlation of driven diffusive systems. To this order, we find that there is no interesting behavior of the time dependence of $\hat{C}^{(0)}(\boldsymbol{k},t)$, which merely exhibits an exponentially decaying form.

The next contribution to $\hat{C}(\boldsymbol{k},t)$ appears at second order in $\bar{\lambda}$. Through a straightforward calculation, we obtain

$$\begin{aligned}\hat{C}^{(2)}(\boldsymbol{k},t) = -2\int_{-\infty}^{\infty}dt'\int\frac{d^2\boldsymbol{k}'}{(2\pi)^2}\frac{\sum_{i=1}^{2}(1-(-1)^i\Delta)k_i'^2}{|\boldsymbol{k}'|^2}\mathrm{e}^{-|\boldsymbol{k}|^2|t-t'|-|\boldsymbol{k}'|^2|t'|-|\boldsymbol{k}-\boldsymbol{k}'|^2|t'|}\\ \left[\frac{\sum_{j=1}^{2}(1-(-1)^j\Delta)k_j^2}{|\boldsymbol{k}|^2}k_1(k_1-k_1')\left((t-t')\frac{t'}{|t'|}+|t-t'|+\frac{1}{|\boldsymbol{k}|^2}\right) - \frac{1}{2}\frac{k_1^2}{|\boldsymbol{k}|^2}\frac{\sum_{j=1}^{2}(1-(-1)^j\Delta)(k_j-k_j')^2}{|\boldsymbol{k}-\boldsymbol{k}'|^2}\right].\end{aligned} \tag{14}$$

(In Sec. A, we present a calculation method to obtain this result.) For this expression, we first perform the integration over $|\boldsymbol{k}'|$ and then consider the t' integration. Next, we carry out the integration over the angle of $\boldsymbol{k}'$. However, this procedure is complicated by the fact that divergences appear in the t' integration. As one example, $\hat{C}^{(2)}(\boldsymbol{k},t)$ includes the term

$$-\frac{1}{4\pi}k_1^2 t\mathrm{e}^{-|\boldsymbol{k}|^2 t}\int_0^t dt'\frac{1}{t'}\mathrm{e}^{|\boldsymbol{k}|^2 t'/2}, \tag{15}$$

where the contribution to the integral around $t' = 0$ yields a logarithmic divergence. Physically, this divergence arises from the interaction between different modes during a very short time interval. However, the model we study is assumed to be appropriate only for describing phenomena over time scales longer than a certain scale τ_{m} in driven diffusive systems [8], and this divergence should not exist in the case that we study a model that correctly describes the phenomena with time scales shorter than τ_{m}. However, here, instead of studying a model in which the microscopic details of behavior on such short time scales are taken into account, we simply introduce a cut-off τ_{m}; that is, the integration range of t' in (14) is replaced by $[-\infty,-\tau_{\mathrm{m}}]\cup[\tau_{\mathrm{m}},\infty]$.

With the cut-off introduced, the term (15) can be regularized as

$$-\frac{1}{4\pi}k_1^2 t\mathrm{e}^{-|\boldsymbol{k}|^2 t}\left[\log\frac{t}{\tau_{\mathrm{m}}}+\int_{\tau_{\mathrm{m}}}^t dt'\frac{1}{t'}(\mathrm{e}^{|\boldsymbol{k}|^2 t'/2}-1)\right]. \tag{16}$$

Here, the first term exhibits a logarithmic divergence as $t/\tau_{\mathrm{m}} \to \infty$, with fixed $|\boldsymbol{k}|^2 t$. Following similar procedures, we can separate all singular terms from $\hat{C}^{(2)}(\boldsymbol{k},t)$, and in each case we obtain a term $\sim \log t/\tau_{\mathrm{m}}$.

Next, we expand $\hat{C}^{(2)}(\boldsymbol{k},t)$ in Δ. Then, to the first order, we obtain

$$\begin{aligned}\hat{C}(\boldsymbol{k},t) &= \hat{C}^{(0)}(\boldsymbol{k},t)\\ &\quad\left[1-(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t)\bar{\lambda}^2\log\frac{t}{\tau_{\mathrm{m}}}\right]\\ &\quad+\bar{\lambda}^2\bar{C}^{(2)}(\boldsymbol{k},t)+o(\bar{\lambda}^2,\Delta),\end{aligned} \tag{17}$$

where $\bar{C}^{(2)}(\boldsymbol{k},t)$ represents the non-singular contribution to $\hat{C}^{(2)}(\boldsymbol{k},t)$, and $c_0(\boldsymbol{k})$ and $c_1(\Delta)$ are given by

$$c_0(\boldsymbol{k}) = \frac{k_1^2}{8\pi|\boldsymbol{k}|^2}\frac{k_1^2-3k_2^2}{|\boldsymbol{k}|^2}, \tag{18}$$

$$c_1(\Delta) = \frac{1}{8\pi}(2-\Delta). \tag{19}$$

(In Sec. B, we present a more detailed explanation of the derivation.) Note that the equal-time correlation $\hat{C}(\boldsymbol{k},0)$ must be obtained as $\lim_{\tau_{\mathrm{m}}\to 0}\hat{C}(\boldsymbol{k},\tau_{\mathrm{m}})$, because the expression (17) is physically sound only for $t \geq \tau_{\mathrm{m}}$.

The bare perturbation result (17) is reliable only for values of t for which $\log t/\tau_{\mathrm{m}}$ is of order unity. Now, employing the RG method demonstrated in Ref. [5], we derive a form of $\hat{C}(\boldsymbol{k},t)$ reliable even for $t \gg \tau_{\mathrm{m}}$. First, we introduce a time scale τ_{M} which can be chosen arbitrarily and define a dimensionless parameter $\mu = \tau_{\mathrm{M}}/\tau_{\mathrm{m}}$. Then, using

$$\log\frac{t}{\tau_{\mathrm{m}}} = \log\frac{t}{\tau_{\mathrm{M}}}+\log\frac{\tau_{\mathrm{M}}}{\tau_{\mathrm{m}}}, \tag{20}$$

we rewrite (17) as

$$\begin{aligned}\hat{C}(\boldsymbol{k},t) &= Z(\mu)\hat{C}^{(0)}(\boldsymbol{k},t)\\ &\quad\left[1-(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t)\bar{\lambda}^2\log\frac{t}{\tau_{\mathrm{M}}}\right]\\ &+ \bar{\lambda}^2\bar{C}^{(2)}(\boldsymbol{k},t)+o(\bar{\lambda}^2,\Delta),\end{aligned} \tag{21}$$

where we have introduced the renormalization constant $Z(\mu)$. Here, the bare perturbation result (17) is equivalent to (21) with

$$Z(\mu) = 1-(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t)\bar{\lambda}^2\log\mu+o(\bar{\lambda}^2,\Delta). \tag{22}$$

Now, we regard (22) as the bare perturbation result for $Z(\mu)$ and calculate the improved perturbation result by using the fact that $\hat{C}(\boldsymbol{k},t)$ does not depend on τ_{M}. That is, differentiating (21) with respect to τ_{M}, we obtain the equation

$$\frac{d\log Z(\mu)}{d\log\mu}+\left(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t\right)\bar{\lambda}^2+o(\bar{\lambda}^2,\Delta)=0, \tag{23}$$

which is referred to as the "renormalization group equation". Solving (23) under the condition

$$Z(\mu=0)=1, \tag{24}$$

we derive

$$Z(\mu) = \mu^{-\left(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t\right)\bar{\lambda}^2+o(\bar{\lambda}^2,\Delta)}, \tag{25}$$

which provides the improved result of (22). Finally, substituting (25) into (21) and setting $\tau_{\mathrm{M}} = t$ (recall that τ_{M} is arbitrary), we obtain the expression

$$\begin{aligned}\hat{C}(\boldsymbol{k},t) &= \hat{C}^{(0)}(\boldsymbol{k},t)\left(\frac{t}{\tau_{\mathrm{m}}}\right)^{-\left(c_0(\boldsymbol{k})\Delta+c_1(\Delta)k_1^2 t\right)\bar{\lambda}^2+o(\bar{\lambda}^2,\Delta)}\\ &+ \bar{\lambda}^2\bar{C}^{(2)}(\boldsymbol{k},t)+o(\bar{\lambda}^2,\Delta),\end{aligned} \tag{26}$$

which may be reliable for all $t \geq \tau_{\mathrm{m}}$.

IV. RESULTS AND REMARKS

From the expression (26), we have the following physically interesting results. First, we note that there is a crossover time $\tau_{\mathrm{c}}(\boldsymbol{k})$ given by

$$\Delta|c_0(\boldsymbol{k})|\bar{\lambda}^2 = |\boldsymbol{k}|^2\tau_{\mathrm{c}}(\boldsymbol{k}). \tag{27}$$

We focus on the small wavenumber regime satisfying $\tau_{\mathrm{c}}(\boldsymbol{k}) \gg \tau_{\mathrm{m}}$. Then, for t satisfying $t/\log(t/\tau_{\mathrm{m}}) \ll \tau_{\mathrm{c}}(\boldsymbol{k})$,

the correlation of density fluctuations takes the power-law form

$$\hat{C}(\boldsymbol{k},t) \simeq \left(1+\Delta\frac{k_1^2-k_2^2}{|\boldsymbol{k}|^2}\right)\left(\frac{t}{\tau_{\rm m}}\right)^{-c_0(\boldsymbol{k})\Delta\bar{\lambda}^2}. \qquad (28)$$

It is important to note that this power-law regime appears only when $\Delta \neq 0$, that is, only when the fluctuation-dissipation relation is violated [see (5)]. We believe that such power-law behavior can be observed in experiments.

In addition to the above result, from (26), we find that the decay rate of the correlation for t satisfying $t/\log(t/\tau_{\rm m}) \gg \tau_{\rm c}(\boldsymbol{k})$ is expressed by

$$-\frac{1}{t}\log\hat{C}(\boldsymbol{k},t) \simeq |\boldsymbol{k}|^2 + c_1(\Delta)k_1^2\bar{\lambda}^2\log t. \qquad (29)$$

This shows that the decay rate of the correlation increases slowly as a function of time. Such enhancement of the decay rate exists even in the case $\Delta = 0$. A similar result was obtained from analysis of the mode coupling equation [3].

The appearance of the singular term $\log t/\tau_{\rm m}$ in the bare perturbation result is the key to obtaining the power-law behavior of the correlation. A similar singular term was treated in Ref. [5] within the framework of the RG method to derive a solution representing anomalous diffusion for a deterministic nonlinear diffusion equation. There are several related works [9, 10] in which such a divergence is treated in a similar way.

The RG analysis given here should not be confused with the method to study the RG flow of system parameters that occurs with the change of the wavenumber scale. For the model under consideration, this type of the RG flow of $(\bar{\lambda},\Delta)$ is investigated in Ref. [7]. With that method, for example, the relevancy of the parameters can be studied, but explicit calculation of the time correlation is not possible.

In conclusion, we calculated the time correlation function (26) for the driven diffusive model (6) with (7). The expression we obtained indicates that a power-law regime appears in the time correlation function if the FDR is violated. In addition to predicting this new type of physical phenomenon, our analysis provides an instructive example for the application of the perturbative RG method.

The authors thank K. Hayashi for her suggesting the perturbative calculation of the time correlation function for the driven diffusive system. This work was supported by a grant from the Ministry of Education, Science, Sports and Culture of Japan (No. 16540337).

APPENDIX A: DERIVATION OF (14)

For an arbitrary function $f(\boldsymbol{x},t)$, we define $\tilde{f}(\boldsymbol{k},\omega)$ as

$$\tilde{f}(\boldsymbol{k},\omega) \equiv \int d^2\boldsymbol{x}dt{\rm e}^{-i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}f(\boldsymbol{x},t). \qquad (A1)$$

Then, the quantity $\tilde{C}(\boldsymbol{k},\omega)$ satisfies

$$(2\pi)^3\delta(\boldsymbol{z}+\boldsymbol{z}')\tilde{C}(z) = \left\langle\tilde{\phi}(\boldsymbol{z})\tilde{\phi}(\boldsymbol{z}')\right\rangle, \qquad (A2)$$

where $\boldsymbol{z} = (\boldsymbol{k},\omega)$. Here, the Fourier transformation of (6) yields

$$\tilde{\phi}(\boldsymbol{z}) = G(\boldsymbol{z})\left[-\sum_{i=1}^2 ik_i\tilde{\xi}_i(\boldsymbol{z}) - \bar{\lambda}ik_1(\tilde{\phi}\circ\tilde{\phi})(\boldsymbol{z})\right], \qquad (A3)$$

with

$$G(\boldsymbol{z}) \equiv \frac{1}{i\omega+\sum_{i=1}^2 k_i^2}, \qquad (A4)$$

where $(\tilde{f}\circ\tilde{g})(\boldsymbol{z})$ denotes the convolution of $\tilde{f}(\boldsymbol{z})$ and $\tilde{g}(\boldsymbol{z})$. From (A3), for $\tilde{\phi}^{(n)}(\boldsymbol{z}), (n=0,1,2,...)$, defined by (11) and (A1), we obtain

$$\tilde{\phi}^{(0)}(\boldsymbol{z}) = G(\boldsymbol{z})\left[-\sum_{i=1}^2 ik_i\tilde{\xi}_i(\boldsymbol{z})\right], \qquad (A5)$$

$$\tilde{\phi}^{(1)}(\boldsymbol{z}) = G(\boldsymbol{z})\left[-ik_1(\tilde{\phi}^{(0)}\circ\tilde{\phi}^{(0)})(\boldsymbol{z})\right], \qquad (A6)$$

$$\tilde{\phi}^{(2)}(\boldsymbol{z}) = G(\boldsymbol{z})\left[-2ik_1(\tilde{\phi}^{(0)}\circ\tilde{\phi}^{(1)})(\boldsymbol{z})\right]. \qquad (A7)$$

We expand $\tilde{C}(\boldsymbol{z})$ in the form

$$\tilde{C}(\boldsymbol{z}) = \tilde{C}^{(0)}(\boldsymbol{z}) + \bar{\lambda}\tilde{C}^{(1)}(\boldsymbol{z}) + \bar{\lambda}^2\tilde{C}^{(2)}(\boldsymbol{z}) + \dots . \qquad (A8)$$

The lowest order contribution of (A8) is calculated as

$$\tilde{C}^{(0)}(\boldsymbol{z}) = 2|G(\boldsymbol{z})|^2\sum_{i=1}^2 k_i^2(1+(-1)^{(i-1)}\Delta). \qquad (A9)$$

Using the inverse Fourier transformation in ω, we obtain (13). It can be easily checked $\tilde{C}^{(1)}(\boldsymbol{z}) = 0$, and $\tilde{C}^{(2)}(\boldsymbol{z})$ is expressed in the form

$$\tilde{C}^{(2)}(\boldsymbol{z}) = \tilde{C}^{(2)}_{\rm I}(\boldsymbol{z}) + \tilde{C}^{(2)}_{\rm II}(\boldsymbol{z}) + \tilde{C}^{(2)}_{\rm III}(\boldsymbol{z}), \qquad (A10)$$

where

$$\tilde{C}^{(2)}_{\mathrm{I}}(\boldsymbol{z}) = 8|G(\boldsymbol{z})|^2 k_1^2 \int d^3\boldsymbol{z}' |G(\boldsymbol{z}-\boldsymbol{z}')|^2 \sum_{i=1}^{2} (k_i - k'_i)^2 (1-(-1)^i\Delta) |G(\boldsymbol{z}')|^2 \sum_{j=1}^{2} k'^2_j (1-(-1)^j\Delta), \tag{A11}$$

$$\tilde{C}^{(2)}_{\mathrm{II}}(\boldsymbol{z}) = 32|G(\boldsymbol{z})|^4 \omega k_1 \sum_{i=1}^{2} k_i^2 (1-(-1)^i\Delta) \int d^3\boldsymbol{z}' |G(\boldsymbol{z}-\boldsymbol{z}')|^2 \sum_{j=1}^{2} (k_j - k'_j)^2 (1-(-1)^j\Delta) |G(\boldsymbol{z}')|^2 \omega' k'_1, \tag{A12}$$

$$\tilde{C}^{(2)}_{\mathrm{III}}(\boldsymbol{z}) = -32|G(\boldsymbol{z})|^4 |\boldsymbol{k}|^2 k_1 \sum_{i=1}^{2} k_i^2 (1-(-1)^i\Delta) \int d^3\boldsymbol{z}' |G(\boldsymbol{z}-\boldsymbol{z}')|^2 \sum_{j=1}^{2} (k_j - k'_j)^2 (1-(-1)^j\Delta) |G(\boldsymbol{z}')|^2 |\boldsymbol{k}'|^2 k'_1. \tag{A13}$$

Note that all the functions $\tilde{C}^{(2)}_{\alpha}(\boldsymbol{z})$, ($\alpha = \mathrm{I}, \mathrm{II}, \mathrm{III}$), take the form

$$\tilde{C}^{(2)}_{\alpha}(\boldsymbol{z}) = \tilde{F}_{\alpha}(\boldsymbol{z})(\tilde{h}_{\alpha} \circ \tilde{\ell}_{\alpha})(\boldsymbol{z}), \tag{A14}$$

where $\tilde{F}_{\alpha}(\boldsymbol{z}), \tilde{h}_{\alpha}(\boldsymbol{z})$ and $\tilde{\ell}_{\alpha}(\boldsymbol{z})$ are determined from (A11)-(A13). Using this form, we can express $\hat{C}^{(2)}_{\alpha}(\boldsymbol{k},t)$ as

$$\hat{C}^{(2)}_{\alpha}(\boldsymbol{k},t) = \int dt' \hat{F}_{\alpha}(\boldsymbol{k}, t-t') \int \frac{d^2\boldsymbol{k}'}{(2\pi)^2} \hat{h}_{\alpha}(\boldsymbol{k}-\boldsymbol{k}', t') \hat{\ell}_{\alpha}(\boldsymbol{k}', t'). \tag{A15}$$

Substituting this into (A10), we obtain (14).

APPENDIX B: DERIVATION OF (17)

We expand $\hat{C}^{(2)}(\boldsymbol{k},t)$ in the form

$$\hat{C}^{(2)}(\boldsymbol{k},t) = \hat{C}^{(2,0)}(\boldsymbol{k},t) + \Delta \hat{C}^{(2,1)}(\boldsymbol{k},t) + o(\Delta). \tag{B1}$$

Through a straightforward calculation, we obtain

$$\hat{C}^{(2,0)}(\boldsymbol{k},t) = \frac{1}{4\pi} k_1^2 \left[\int_0^t dt' \mathrm{e}^{|\boldsymbol{k}|^2 t'/2} - t \int_0^t dt' \frac{1}{t'} \mathrm{e}^{|\boldsymbol{k}|^2 t'/2} \right]. \tag{B2}$$

In order to calculate $\hat{C}^{(2,1)}(\boldsymbol{k},t)$, we extract terms proportional to Δ from (14). The obtained expression becomes

$$\begin{aligned}\hat{C}^{(2,1)}(\boldsymbol{k},t) =& -4 \int_0^t dt' k_1 (t-t') \mathrm{e}^{-|\boldsymbol{k}|^2(t-t')} \int \frac{d^2\boldsymbol{k}'}{(2\pi)^2} \left[\frac{k'^2_1 - k'^2_2}{|\boldsymbol{k}'|^2} + \frac{k_1^2 - k_2^2}{|\boldsymbol{k}|^2} \right] (k_1 - k'_1) \mathrm{e}^{-[|\boldsymbol{k}-\boldsymbol{k}'|^2 + |\boldsymbol{k}'|^2]t'} \\ & + 2\frac{k_1^2}{|\boldsymbol{k}|^2} \int dt' \mathrm{e}^{-|\boldsymbol{k}|^2|t-t'|} \int \frac{d^2\boldsymbol{k}'}{(2\pi)^2} \frac{k'^2_1 - k'^2_2}{|\boldsymbol{k}'|^2} \mathrm{e}^{-[|\boldsymbol{k}-\boldsymbol{k}'|^2 + |\boldsymbol{k}'|^2]|t'|} \\ & - 2\frac{k_1}{|\boldsymbol{k}|^2} \int dt' \mathrm{e}^{-|\boldsymbol{k}|^2|t-t'|} \int \frac{d^2\boldsymbol{k}'}{(2\pi)^2} \left[\frac{k'^2_1 - k'^2_2}{|\boldsymbol{k}'|^2} + \frac{k_1^2 - k_2^2}{|\boldsymbol{k}|^2} \right] (k_1 - k'_1) \mathrm{e}^{-[|\boldsymbol{k}-\boldsymbol{k}'|^2 + |\boldsymbol{k}'|^2]|t'|}. \end{aligned} \tag{B3}$$

We first evaluate the Gauss integrals in $|\boldsymbol{k}'|$ and perform the t' integrals with picking up singular terms. Then, $\hat{C}^{(2,1)}(\boldsymbol{k},t)$ is obtained as

$$\begin{aligned}\hat{C}^{(2,1)}(\boldsymbol{k},t) =& -\frac{1}{8\pi} \mathrm{e}^{-|\boldsymbol{k}|^2 t} k_1^2 t \left(\frac{k_1^2 - 3k_2^2}{|\boldsymbol{k}|^2} \right) \ln \frac{t}{\tau_{\mathrm{m}}} \\ & - \frac{1}{8\pi} \mathrm{e}^{-|\boldsymbol{k}|^2 t} \frac{k_1^2}{|\boldsymbol{k}|^2} \left(\frac{k_1^2 - 3k_2^2}{|\boldsymbol{k}|^2} \right) \ln \frac{t}{\tau_{\mathrm{m}}} \\ & + (\text{non-singular term}). \end{aligned} \tag{B4}$$

Combining (B2) and (B4) with (13), we finally obtain (17) with (18) and (19).

[1] Y. Pomeau and P. Résibois, Physics Reports **19**, 63, (1975).

[2] P. T. Korda, M. B. Taylor and D. G. Grier, Phys. Rev.

Lett. **89** 128301, (2002).
[3] H. van Beijeren, R. Kutner and H. Spohn, Phys. Rev. Lett. **54** 2026, (1985).
[4] G. Grinstein, D. Lee and S. Sachdev, Phys. Rev. Lett. **64**, 1927, (1990).
[5] N. Goldenfeld, O. Martin, Y. Oono and F. Liu, Phys. Rev. Lett. **64**, 1361 (1990).
[6] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Springer, Berlin, 1991).
[7] R. K. P. Zia and B. Schmittmann, Z. Phys. B. **97**, 327, (1995).
[8] Our model is believed to correspond to a coarse-grained model of a driven lattice gas. See, for example, B. Schmittman and R. K. P. Zia, *Phase transitions and Critical Phenomena Volume 17, Statistical Mechanics of Driven Diffusive Systems*, (Academic Press, 1995). It has been found in numerical experiments on driven lattice gases that such a time scale $\tau_{\rm m}$ does indeed appear.
[9] H. K. Janssen, B. Schaub and B. Schmittmann, Z. Phys. B. **73**, 539, (1989).
[10] P. Calabres and A. Gambassi, cond-mat/0406289.